

Lattice Gross-Neveu model with domain-wall fermions*

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We investigate the two-dimensional lattice Gross-Neveu model, using the domain-wall fermion formulation, as a toy model of lattice QCD. We study features of the phase diagram related to the mechanism of chiral symmetry restoration, and find that the parity-broken phase (Aoki phase) exists for finite extent in the extra dimension (N_s). We also find that $O(a)$ scaling violation terms vanishes in the limit of $N_s \rightarrow \infty$.

1. Introduction

Defining chiral fermions on the lattice has been one of long-standing problems in lattice field theories. Several years ago, domain-wall fermion [1,2] has been proposed as a new formulation of lattice chiral fermion. This formulation considers Wilson fermion in $D+1$ dimensions with the free boundary condition in the extra dimension of a size N_s , or equivalently, N_s -flavored Wilson fermions with flavor mixing. In the limit $N_s \rightarrow \infty$, the spectrum of free domain-wall fermion (DWF) contains massless modes at the edges in the extra dimension. The massless modes are stable under perturbation from weak gauge fields.

While there have been numerical simulations of lattice QCD with DWF (DWQCD), some non-perturbative issues, in particular existence of the parity-broken phase (Aoki phase)[3] and necessity of fine tuning of couplings to restore chiral symmetry for finite N_s , have not been clarified. In order to answer these questions, we investigate the two-dimensional lattice Gross-Neveu model using the DWF formalism (DWGN) [4] as a test of DWQCD.

2. Action and Effective potential

We propose the following action for DWGN:

$$S = S_{free} + a^2 \sum_n \bar{q}(n) \{ \sigma(n) + i\gamma_5 \Pi(n) \} q(n)$$

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$$+ a^2 \sum_n \left[\frac{N}{2g_\sigma^2} \{ \sigma(n) - m_f \}^2 + \frac{N}{2g_\pi^2} \Pi(n)^2 \right]. \quad (1)$$

The auxiliary fields are related to fermion condensates as $\sigma(n) = m_f - \frac{g_\sigma^2}{N} \bar{q}(n)q(n)$ and $\Pi(n) = -\frac{g_\pi^2}{N} \bar{q}(n)i\gamma_5 q(n)$. The important point in this action is that the interaction terms are constructed from the edge state: $q(n) = P_R \psi(n, s=1) + P_L \psi(n, s=N_s)$.

We attempt to solve DWGN analytically in the large N limit. The effective potential can be calculated by the technique of the propagator matrix[5,6], which yields

$$V_{eff} = \frac{1}{2g_\sigma^2} (\sigma - m_f)^2 + \frac{1}{2g_\pi^2} \Pi^2 - I, \quad (2)$$

$$I = \int_p \ln [F a^2 (\sigma^2 + \Pi^2) + G a \sigma + H], \quad (3)$$

where $\int_p = \int_{-\pi/a}^{\pi/a} \frac{d^2 p}{(2\pi)^2}$. The explicit form of the functions F, G and H are described in Ref.[4]. Let us note that the term $G a \sigma$ explicitly breaks chiral symmetry.

3. Existence of Aoki phase

The phase diagram of the Wilson fermion action has a region of spontaneously broken parity-flavor symmetry (Aoki phase), which plays an important role for controlling restoration of chiral symmetry in the continuum limit[3,7]. Since DWF is an extension of the Wilson fermion formalism, we wish to examine if Aoki phase exists

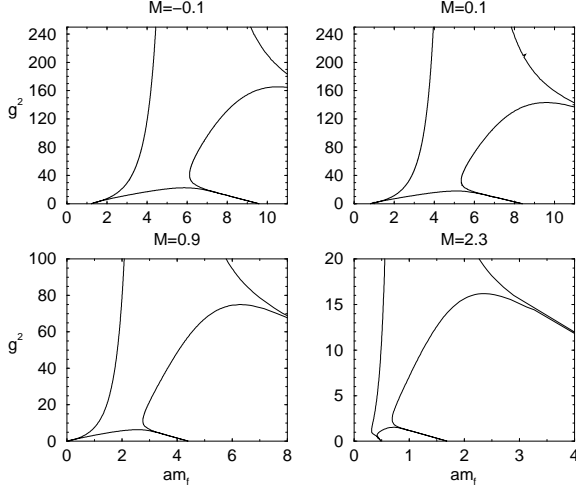


Figure 1. Phase diagram at $N_s = 2$ on (am_f, g^2) plane for several values of M .

for DWGN. In this section, we set the coupling constants as $g_\sigma^2 = g_\pi^2 = g^2$.

Let us first consider the case of $N_s = \infty$. Since F and H dominates over the term G in this limit, the effective potential becomes

$$I(\sigma, \Pi) = \int_p \ln [Fa^2(\sigma^2 + \Pi^2) + H] . \quad (4)$$

The $O(a)$ term, which breaks chiral symmetry, is absent, and hence the model has exact chiral symmetry. Thus pion becomes massless without fine tuning even for finite lattice spacings and for arbitrary strong coupling.

Next we consider the case of finite N_s . It is expected that Aoki phase exists in this case since the $N_s = 1$ DWGN is equivalent to GN model with Wilson fermion[7].

The solution of the gap equation marking the phase boundary is illustrated for several values of M with a fixed size $N_s = 2$ in Fig.1. As is summarized in a schematic diagram in Fig.2, we find that (i) Aoki phase exists inside the boundary and this boundary forms cusps toward $g^2 \rightarrow 0$, (ii) the Aoki phase always appears in the $m_f > 0$ region for even N_s (in the conventional choice of sign in domain-wall literatures, this corresponds to $m_f < 0$), (iii) pion becomes massless on the boundary. The last point means that pion mass

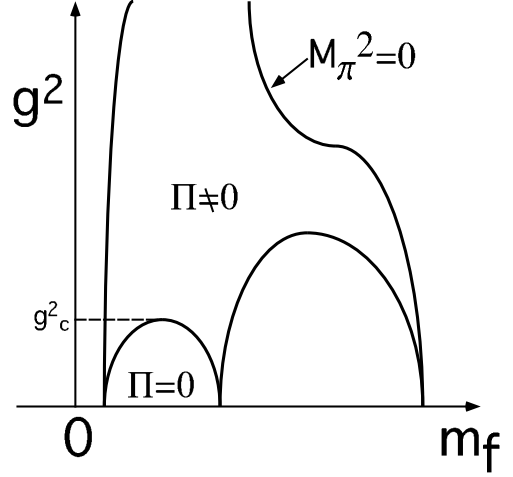


Figure 2. Schematic phase diagram on (m_f, g^2) plane for $N_s = \text{even}$.

does not vanish at $m_f = 0$ for finite lattice spacing. Hence a fine tuning is needed to obtain massless pion.

In Fig.3 we show the N_s dependence of the phase boundary for $N_s = 2, 4$ and 6 at $M = 0.9$. We find that (i) the Aoki phase shrinks exponentially with N_s , (ii) the first “finger” on the left approaches $m_f = 0$, while other “fingers” move toward $m_f = \infty$, (iii) the critical value g_c (see Fig.2), below which the cusp structure (“fingers”) appears, increases exponentially with N_s , in contrast to the case of Wilson fermion[7]. These features mean that the area of the normal phase becomes wide with increasing N_s .

As seen from above, DWF for finite N_s represents an improved Wilson fermion.

4. $O(a)$ scaling violation ($a \rightarrow 0$)

We study the mechanism of chiral symmetry restoration in the continuum limit. In particular, we wish to understand scaling violation for finite and infinite N_s .

The effective potential in the continuum limit is obtained after some calculation:

$$V_{eff} = -m\sigma_R + \frac{1}{4\pi} (\sigma_R^2 + \Pi_R^2) \ln \frac{\sigma_R^2 + \Pi_R^2}{e\Lambda^2} . \quad (5)$$

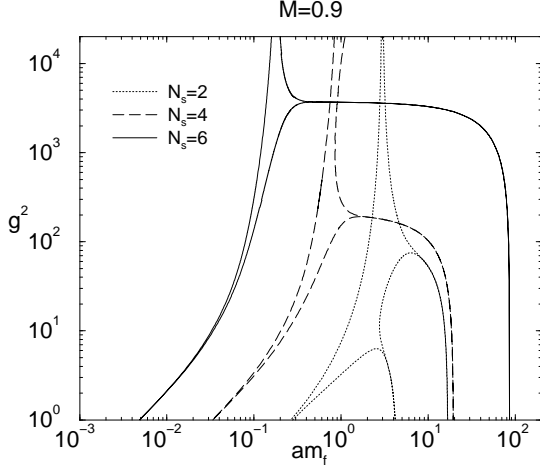


Figure 3. N_s dependence of the phase boundary

Here $\sigma_R = f_M \left\{ \sigma - \frac{(1-M)^{N_s}}{a} \right\}$ and $\Pi_R = f_M \Pi$ with $f_M = \frac{M(2-M)}{1 - (1-M)^{2N_s}}$, which is the normalization factor of the edge state “ $q(n)$ ” for finite N_s . In order to obtain (5), which agrees with the continuum result, we need to impose the following scaling relations:

$$\frac{1}{2g_\sigma^2} - \hat{C}_0 + C_2 = \frac{f_M^2}{4\pi} \ln \frac{1}{a^2 \Lambda^2}, \quad (6)$$

$$\frac{1}{2g_\pi^2} - \hat{C}_0 = \frac{f_M^2}{4\pi} \ln \frac{1}{a^2 \Lambda^2}, \quad (7)$$

$$\frac{1}{f_M} \left(\frac{m_f}{g_\sigma^2} - \frac{(1-M)^{N_s}}{g_\sigma^2 a} + \frac{C_1}{a} \right) = m. \quad (8)$$

These relations are the same with those found for the Wilson fermion case [7]. Therefore a fine tuning is needed to restore chiral symmetry.

Figure 4 shows σ/Λ as a function of $a\Lambda$ for $M = 0.9$, using the Wilson-like scaling relations in (6-8). We find that $O(a)$ scaling violation is large at $N_s = 2, 3$. However, the magnitude of $O(a)$ scaling violation diminishes exponentially as N_s increases. In fact the scaling curve almost exactly follows the $O(a^2)$ behavior for $N_s \geq 8$.

5. Conclusions

We have investigated the two-dimensional DWGN model in detail as a test of DWQCD.

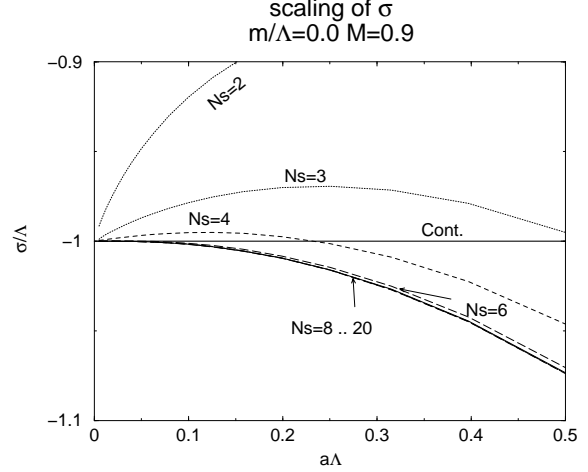


Figure 4. Scaling of σ/Λ as a function of $a\Lambda$

When the size of the extra dimension $N_s = \infty$, the model has chiral symmetry for finite lattice spacing, and Aoki phase does not exist. On the other hand, in the case of finite N_s , Aoki phase does exist and a fine tuning is needed to restore chiral symmetry in the continuum limit. However, the $O(a)$ scaling violation that gives rise to this behavior vanishes exponentially fast as N_s is increased so that it is negligible in practice for $N_s = O(10)$.

While the GN model does not have gauge fields and quantum fluctuations are absent in the large N limit, it is expected that the results obtained in this work provide instructive and systematic information for DWQCD simulations.

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